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Problem 1

(i)

a) $x' = x^2 + 2x = f(x)$

$f(x) = 0$

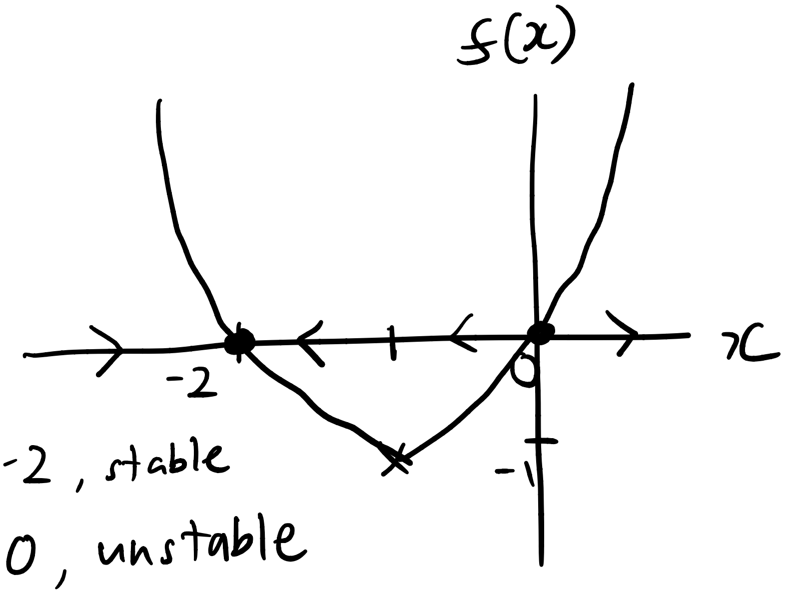
$\Rightarrow x^2 + 2x = 0$

$x(x+2) = 0$

$x = -2, 0$

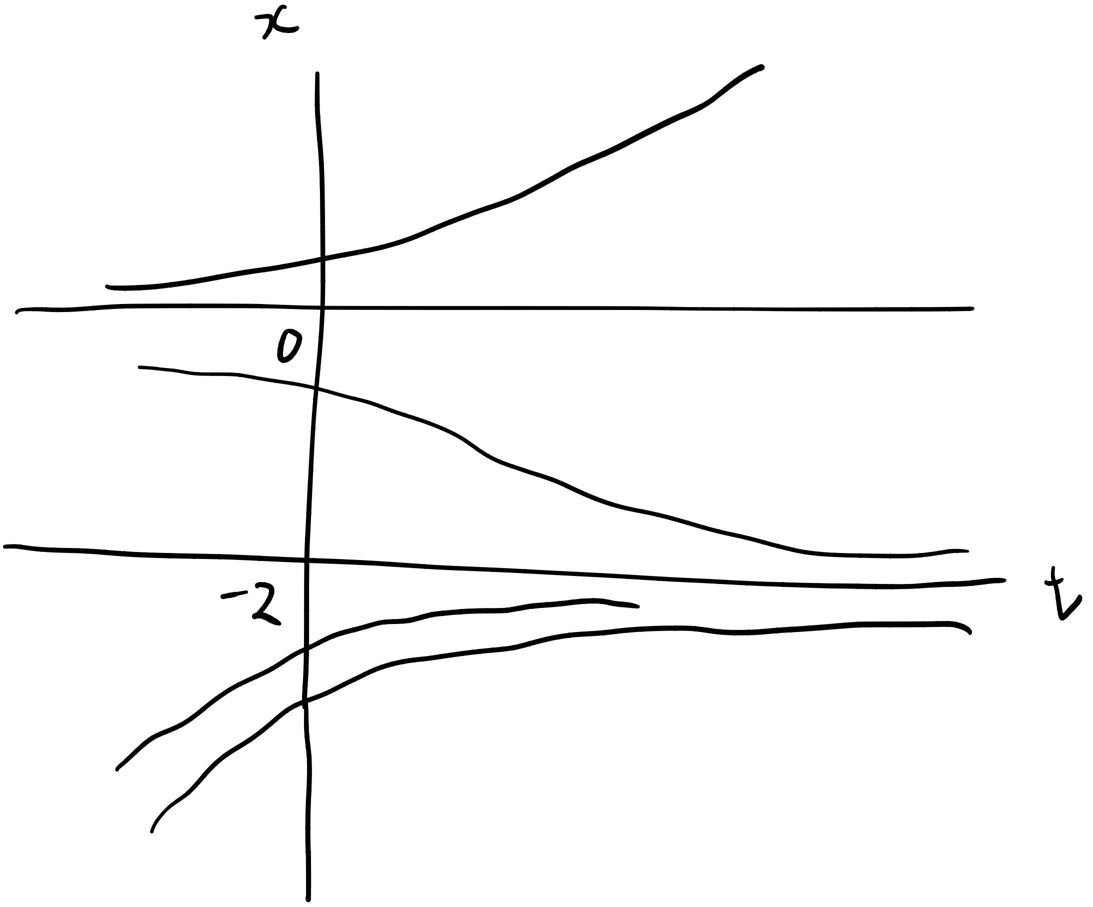
$f(x) = x^2 + 2x + 1^2 - 1^2$

$= (x+1)^2 - 1$



Critical point: $x = -2$, stable

$x = 0$, unstable



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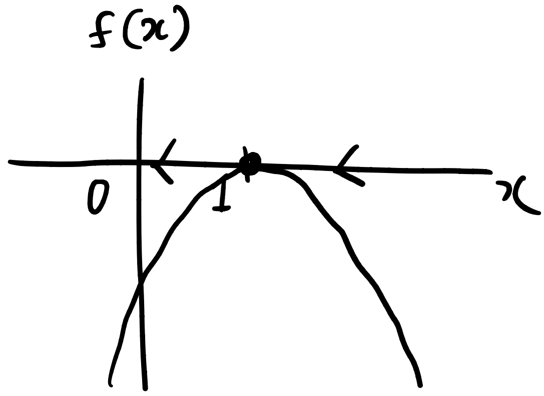
b)

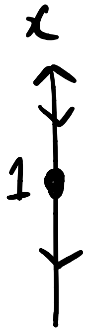
(i) $x' = -(x-1)^2$

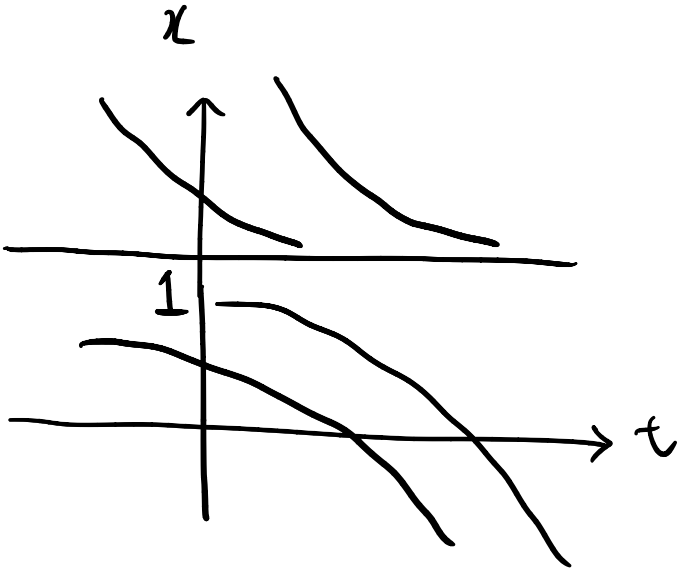
$f(x) = -(x-1)^2$



$\therefore$ semi-stable

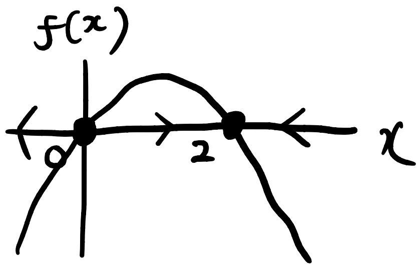
(ii)





c) $x' = 2x - x^2$

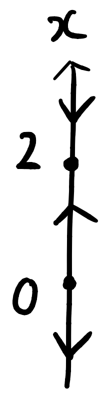
(i) $f(x) = -x(x-2) = 0$

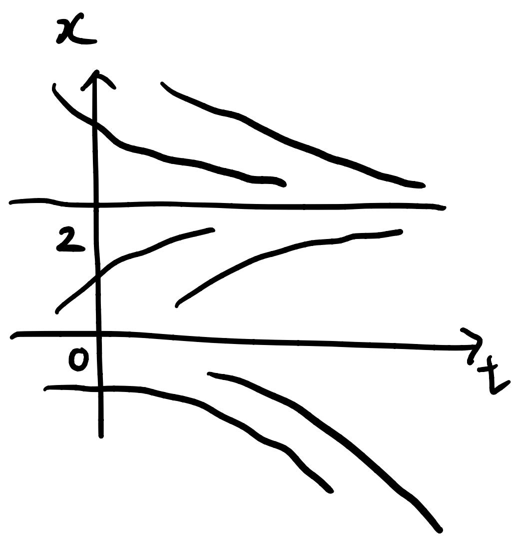


$\therefore$ $x = 0$, unstable

$x = 2$, stable

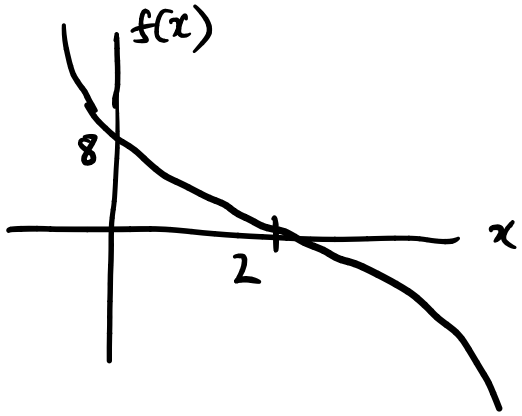
(ii)



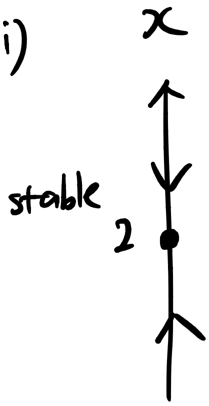


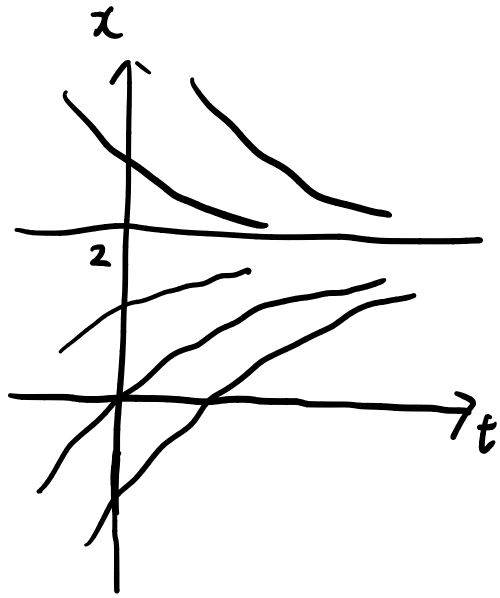
d) $x' = (2-x)^3$

(i) $f(x) = -(x-2)^3$



(ii)





Problem 2

$$\dot{x} + 2x = 1$$

a) (i)

$$\frac{dx}{dt} = 1 - 2x$$

$$\int \frac{1}{1-2x} \frac{dx}{dt} dt = \int 1 \, dt$$

$$\frac{\ln|1-2x|}{-2} = t + C_1$$

$$\ln|1-2x| = -2t - 2C_1$$

$$1 - 2x = \pm e^{-2t} \cdot e^{-2C_1}$$

$$-2x = C_2 e^{-2t} - 1$$

$$x = Ce^{-2t} + \frac{1}{2}$$

(ii)

$$u = e^{\int 2 dt} = e^{2t}$$

$$u\dot{x} + 2ux = u$$

$$\Rightarrow \frac{d}{dt}(ux) = u\dot{x} + 2ux$$

$$\Rightarrow \int \frac{d}{dt}(e^{2t}x) = \int e^{2t}$$

$$e^{2t}x = \frac{e^{2t}}{2} + C$$

$$x = \frac{1}{2} + Ce^{-2t}$$

(iii)

$$\dot{x} + 2x = e^{0t}$$

Guess $x = Ae^{0t}$.

$$\Rightarrow 0 \cdot Ae^{0t} + 2Ae^{0t} = e^{0t}$$

$$2A = 1$$

$$A = \frac{1}{2}$$

$$\therefore x = \frac{1}{2}e^{0t} = \frac{1}{2}$$

$$\dot{x} + 2x = 0$$

$$\dot{x} = -2x$$

$$\int \frac{1}{x} \dot{x} \, dt = -2 \int 1 \, dt$$

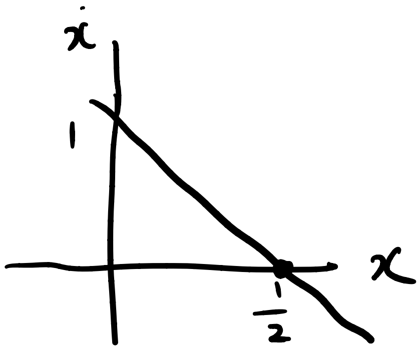
$$\ln|x| = -2t + C_1$$

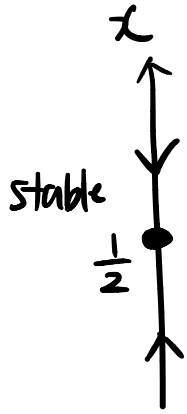
$$x = Ce^{-2t}$$

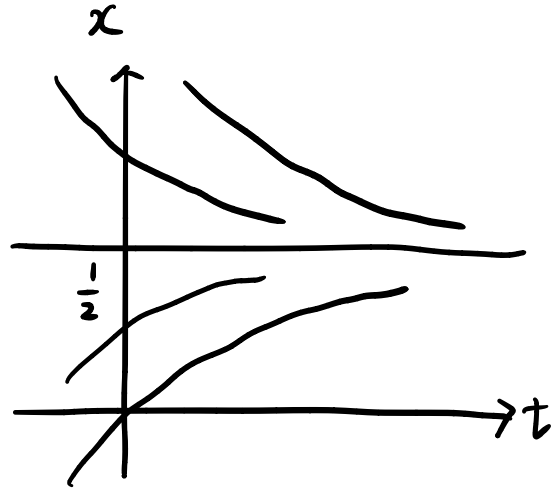
By Principle of Superposition,

$$\Rightarrow x = \frac{1}{2} + Ce^{-2t}$$

b) $\dot{x} = 1 - 2x$







c) $x(0) = 0$, $h = \frac{1}{3}$

$x(0) = 0$

$\Rightarrow 0 = C + \frac{1}{2}$

$C = -\frac{1}{2}$

$\Rightarrow x = \frac{1}{2}(1 - e^{-2t})$

$x(1) = 0.432$

$x\left(\frac{1}{3}\right) = x_0 + hx_0'$

$= 0 + \frac{1}{3}(1 - 2(0))$

$= \frac{1}{3}$

$x\left(\frac{2}{3}\right) = x_1 + hx_1'$

$= \frac{1}{3} + \frac{1}{3}\left(1 - 2\left(\frac{1}{3}\right)\right)$

$= \frac{1}{3} + \frac{1}{9}$

$= \frac{4}{9}$

$x(1) = x_2 + hx_2'$

$= \frac{4}{9} + \frac{1}{3}\left(1 - 2\left(\frac{4}{9}\right)\right)$

$= \frac{4}{9} + \frac{1}{3}\left(\frac{1}{9}\right)$

$= \frac{12}{27} + \frac{1}{27}$

$= \frac{13}{27}$ $\quad \therefore x(1) = \frac{13}{27} = 0.481$